LIFTING-LINE THEORY AS A SINGULAR-PERTURBATION PROBLEM

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The method of matched asymptotic expansions, recently developed for treating singular-perturbation problems, is applied to the flat unswept lifting wing of high aspect ratio. This yields a simplified equivalent of Prandtl's lifting-line theory, with the solution of an integral equation replaced by quadratures. The next approximation is calculated in general terms. Specific application is made to cusped, lenticular, elliptic and rectangular planforms, and comparison drawn where possible with previous work. Additional non-uniformities at tips and other discontinuities are described, and procedures outlined for their correction.

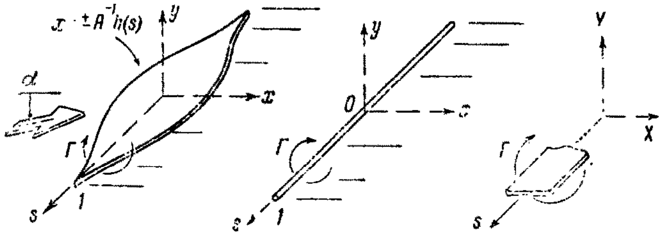
A perturbation problem is called singular if a straightforward expansion is invalid in some part of the field of interest. We can anticipate such non-uniformity in fluid mechanics whenever a problem contains two characteristic lengths and we approximate for small values of their ratio.

The prototype of singular-perturbation problems is Prandtl's boundary-layer theory, where we seek a first approximation for small ratios of the viscous length ν/V to a typical geometric length. A systematic procedure for calculating higher approximations in such problems has been developed at California Institute of Technology by Lagerstrom, Kaplun and Cole [1] and [2]. This method of matched asymptotic expansions has been applied to a variety of viscous flows.

Recently at Stanford University we have applied the method also to a number of inviscid flows. Yakura [3] has treated the entropy layer produced by slightly blunting a pointed body at hypersonic speed. Munson (unpublished) is exploring the vortical layer on an inclined cone at supersonic speed. The present paper is devoted to a re-examination of the classical theory of high-aspect-ratio wings in subsonic flow.

In contrast to his boundary-layer theory, Prandtl's lifting-line theory was only recently recognized as a singular-perturbation problem. It yields an asymptotic solution for vanishing ratio of chord to span. The role of these two disparate reference lengths was recognized by Friedrichs [4], who reproduced Prandtl's integral equation in a pioneering application of the method of matched expansions. Here we show how systematic use of the method simplifies that result by reducing it to quadratures, and provides the next approximation.

1. Matched asymptotic expansions. Consider for simplicity a flat wing of zero thickness, whose planform is symmetric in the chordwise as well as the spanwise direction (Fig. 1*a*); also, let the angle of attack α be so small that only linear terms are significant. Take the free-stream speed and the semispan as units of velocity and length. Let the wing lie in the plane $y = 0$, to which the stream is inclined from below.



a *b* *c*

Fig. 1

It is convenient to describe the planform by $x = \pm A^{-1} h(s)$, where A is the aspect ratio, s is the spanwise coordinate, and the half-chord h is a function of order unity. We suppose temporarily that h is as smooth as may be necessary; the consequences of ignoring this restriction at blunt tips will be faced later.

The full problem for the velocity potential φ is

$$\varphi_{xx} + \varphi_{yy} + \varphi_{ss} = 0 \quad \text{(equation of motion)} \tag{1.1.1}$$

$$\varphi_y = 0 \quad \text{for } y=0,\ |x| \leqslant A^{-1}h(s),\ |s| \leqslant 1 \quad \text{(tangency condition)} \tag{1.1.2}$$

$$\varphi \sim x + \alpha y \quad \text{(upstream condition)} \tag{1.1.3}$$

$$|\varphi_x|,\ |\varphi_y|,\ |\varphi_s| < \infty, \quad \text{for } y=0,\ x=A^{-1}h(s),\ |s|<1 \tag{1.1.4}$$

(Kutta-Zhukovskii condition)

1.1. The outer limit. Now let the aspect ratio A become infinite with x, y, s fixed. We call this the outer limit process – and x, y, s, φ outer variables – because the major dimension is used as reference length. The wing shrinks to a line (Fig. 1*b*), in which are concentrated all the singularities that may be used to represent it.

It is clear that in this limit the disturbances produced by the wing vanish like A^{-1}. Thus the outer limit is simply the uniform stream. If we keep small perturbations of order A^{-1} it is plausible (and will be confirmed later by matching) that the complete system of singularities is approximated by a bound line vortex of unknown variable circulation

$$\Gamma(s; A) \sim A^{-1}\gamma_2(s) \tag{1.2}$$

Associated with the spanwise variation are free vortices that trail backward approximately in the free-stream direction but may, for small α, be taken to lie in the plane of the wing $y = 0$.

This is the familiar vortex system of Prandtl's lifting-line theory. Its velocity potential is

$$\varphi \sim (x + \alpha y) + \frac{1}{4\pi} A^{-1} \int_{-1}^{1} \frac{y\gamma_2(\sigma)}{y^2 + (s-\sigma)^2} \left[1 + \frac{x}{\sqrt{x^2 + y^2 + (s-\sigma)^2}}\right] d\sigma \qquad (1.3)$$

1.2. The inner limit   Unfortunately it is now too late to find the bound vorticity γ_2. It could have been determined by imposing the Kutta-Zhukovskii condition at the trailing edge; but such detail has been lost in the outer limiting process. To recover that detail, we magnify the coordinates so that they are referred to a typical chord rather than the semi-span. We correspondingly magnify the velocity potential, so that all variables are of order unity near the wing. Thus we introduce inner variables, denoted by capital letters, by setting

$$\varphi = A^{-1}\Phi(X, Y, s), \qquad X = Ax, \qquad Y = Ay \qquad (1.4)$$

Then the full problem (1.1) becomes

$$\Phi_{XX} + \Phi_{YY} + A^{-2}\Phi_{ss} = 0 \qquad (1.5.1)$$

$$\Phi_Y = 0 \quad \text{at} \quad Y = 0,\ |X| \leqslant h(s),\ |s| \leqslant 1 \qquad (1.5.2)$$

$$\Phi \sim X + \alpha Y \qquad \text{(upstream)} \qquad (1.5.3)$$

$$|\Phi_X|,\ |\Phi_Y|,\ |\Phi_s| < \infty \quad \text{at} \quad Y = 0,\ X = h(s),\ |s| \leqslant 1 \qquad (1.5.4)$$

Setting $A = \infty$ gives the problem for the inner limit (Fig. 1c). It is evidently that of plane flow past a flat plate, the spanwise coordinate s appearing only parametrically in $h(s)$. The velocity potential is

$$\Phi_1 = X + \alpha\, \mathrm{Im}\, [(Z^2 - h^2)^{1/2} - h \cosh^{-1}(Z/h)], \qquad Z = X + iY \qquad (1.6)$$

We were not strictly justified here in imposing the free-stream condition (1.5.3), because the inner solution is intended to hold only near the wing. However, the applicability of that condition is justified by matching the inner and outer solutions, using the restricted matching principle [1]:

m-term inner expansion of (p-term outer expansion) = p-term outer expansion of (m-term inner expansion) (1.7)

It would suffice to take $m = p = 1$; but in the next step we will need $m = 1$ and $p = 2$. We therefore calculate at once the 2-term outer expansion of the inner limit. This is found by rewriting $\varphi = A^{-1}\Phi_1$ in outer variables, expanding for large A, and keeping secondary as well as leading terms

$$\text{2-outer (1-inner)} \qquad \varphi = (x + \alpha y) - \frac{\alpha h(s)}{A} \tan^{-1} \frac{y}{x} \qquad (1.8)$$

The first term justifies our use of the upstream condition. The second term is the plane potential due to a vortex at the origin, of circulation

$$\Gamma = \Gamma_\infty = 2\pi\alpha A^{-1} h(s) \tag{1.9}$$

1.3. Second approximation. To proceed formally, we would seek solution of the outer equations whose inner limit was (1.8). However, we have already constructed such a solution by physical reasoning, as the lifting-line potential (1.3). It remains only to determine the vorticity γ_2 appearing there by matching with the inner limit. The result is evident physically from the fact that the circulation is the same about any two curves enclosing the same vortex lines (Fig. 1). Hence (1.2) and (1.9) can be equated, giving

$$\gamma_2(s) = 2\pi\alpha h(s) \tag{1.10}$$

Then substituting into (1.3) yields the 2-term outer expansion

$$\varphi \sim (x + \alpha y) + \frac{\alpha}{2A}\int_{-1}^{1} \frac{y h(\sigma)}{y^2 + (s-\sigma)^2}\left[1 + \frac{x}{\sqrt{x^2 + y^2 (s-\sigma)^2}}\right] d\sigma \tag{1.11.1}$$

as $A \to \infty$ with x, y, s fixed

In the next step we will need the 2-term inner expansion of this result. Divergent integrals are avoided by differentiating with respect to s and integrating under the integral to obtain the alternative form (1.11.2)

$$\varphi \sim (x + \alpha y) - \frac{\alpha}{2A}\frac{\partial}{\partial s}\int_{-1}^{1} h(\sigma)\left[\tan^{-1}\frac{y}{s-\sigma} + \tan^{-1}\frac{y}{x}\frac{\sqrt{x^2+y^2+(s-\sigma)^2}}{s-\sigma}\right] d\sigma$$

This strategem, which will be repeated later, is familiar from slender-body theory [5]. Now introducing inner variables and expanding gives

$$\text{2-inner (2-outer)} \quad \varphi = \frac{1}{A}\left[X + \alpha Y - \alpha h(s) \tan^{-1}\frac{Y}{X}\right] - \frac{1}{2}\frac{\alpha}{A^2} Y \int_{-1}^{1} \frac{h'(\sigma)\, d\sigma}{s-\sigma} \tag{1.12}$$

The Cauchy principal value of the integral is indicated for the usual reason. The first term agrees with (1.8) to confirm our physical matching argument.

We now return to the inner expansion and seek a second approximation. Equation (1.5.1) would suggest a correction of relative order A^{-2}, but matching with (1.12) shows that a term of relative order A^{-1} intervenes. We therefore make the inner expansion

$$\varphi \sim A^{-1}\Phi_1(X, Y, s) + A^{-2}\Phi_2(X, Y, s) + \dots \tag{1.13}$$

as $A \to \infty$ with X, Y, s fixed

Substituting into the full problem (1.5) shows that Φ_2, like Φ_1, satisfies Laplace's equation in the X-Y plane with zero velocity normal to the wing. In matching with (1.12), the coefficients of Y indicate that Φ_2 is simply the result of reducing the angle of attack in the flat-plate solution (1.6) from its geometric value α to an effective value α_e, where

$$\frac{\alpha_e}{\alpha} = 1 - \frac{1}{2A}\int_{-1}^{1}\frac{h'(\sigma)\,d\sigma}{s-\sigma} \tag{1.14}$$

This is a familiar result of lifting-line theory [6]; the trailing-vortex system induces downwash velocities near the wing that are constant across the chord at each spanwise station, and so act to reduce the apparent angle of attack of that section. However, our procedure has reduced the calculation to a quadrature, rather than the solution of an integral equation, by taking advantage of the fact that the downwash angle is small compared with the geometric angle of attack.

The 2-term inner expansion is given by (1.6) with α replaced by α_e from (1.14)

$$\varphi \sim \frac{X}{A} + \frac{\alpha}{A}\left[1 - \frac{1}{2A}\int_{-1}^{1}\frac{h'(\sigma)d\sigma}{s-\sigma}\right] \operatorname{Im}\left[(Z^2 - h^2)^{1/2} - h\cosh^{-1}\left(\frac{Z}{h}\right)\right] \tag{1.15}$$

1.4. Third outer approximation. Before comparing our solution with the classical theory, we persevere to find the next approximation. Introducing outer variables into (1.15) and expanding gives

3-outer (2-inner)
$$\varphi = x + \alpha\left(1 - \frac{1}{2A}\int_{-1}^{1}\frac{h'(\sigma)d\sigma}{s-\sigma}\right) \times \left[y - \frac{h(s)}{A}\tan^{-1}\frac{y}{x}\right] + \frac{\alpha}{2A^2}h^2\frac{y}{x^2+y^2} \tag{1.16}$$

Repeating the previous argument shows that the circulation of the bound vortices has been refined from (1.9) to

$$\Gamma = 2\pi\alpha A^{-1}h(s)\left[1 - \frac{1}{2A}\int_{-1}^{1}\frac{h'(\sigma)d\sigma}{s-\sigma}\right] = 2\pi\alpha A^{-1}\left[h(s) + \frac{1}{A}h_2(s)\right] \tag{1.17}$$

Here we introduce

$$h_2(s) = -\frac{1}{2}h(s)\int_{-1}^{1}\frac{h'(\sigma)d\sigma}{s-\sigma} \tag{1.18}$$

representing the second-order change in half-chord for a fictitious wing having in the first approximation the same lift as the actual wing has in the second approximation.

In addition to this change in vorticity, the last term in (1.16) shows the appearance at this stage of the next higher singularity in the loaded line. We call this a divortex (although it may be regarded as a dipole with vertical axis) to emphasize that it is the x-derivative of a vortex, representing physically the first moment of the distributed vorticity of the lifting surface. The divortex strength is $\Delta \sim A^{-2}\delta_3$, where

$$\delta_3(s) = \pi\alpha h^2(s) \tag{1.19}$$

The potential of a spanwise distribution of divortices is just the x-derivative of that (1.3) for vortices. Thus the 3-term outer expansion is found to be

$$\varphi \sim (x+\alpha y) - \frac{\alpha}{2A}\frac{\partial}{\partial s}\int_{-1}^{1}\left[h(\sigma)+\frac{h_2(\sigma)}{A}\right]\left[\tan^{-1}\frac{y}{s-\sigma}+\right. \tag{1.20}$$

$$\left.+\tan^{-1}\frac{y}{x}\frac{\sqrt{x^2+y^2+(s-\sigma)^2}}{s-\sigma}\right]d\sigma+\frac{\alpha}{4A^2}\frac{y}{x^2+y^2}\frac{\partial}{\partial s}\int_{-1}^{1}h^2(\sigma)\frac{(s-\sigma)\,d\sigma}{\sqrt{x^2+y^2+(s-\sigma)^2}}$$

As before, divergent integrals in the inner expansion are avoided in the integrals of order A^{-2} by differentiating twice more with respect to s and integrating under the integral sign. Then some computation gives the 3-term inner expansion of (1.20)

3-inner (3-outer)
$$\varphi = \frac{1}{A}\left[X+\alpha\left\{Y-h(s)\tan^{-1}\frac{Y}{X}+\right.\right.$$

$$\left.\left.+\frac{1}{2}h^2(s)\frac{Y}{X^2+Y^2}\right\}\right]+\frac{\alpha}{2A^2}\left[h(s)\tan^{-1}\frac{Y}{X}-Y\right]\int_{-1}^{1}\frac{h'(\sigma)\,d\sigma}{s-\sigma}+$$

$$+\frac{\log A\,\alpha}{2A^3}\left[XYh''(s)+Y(hh')'\right]+\frac{\alpha}{2A^3}\left[-Y\frac{d}{ds}\int_{-1}^{1}\frac{h_2(\sigma)d\sigma}{s-\sigma}+\right.$$

$$+Y\left(\log\frac{2}{\sqrt{X^2+Y^2}}+\frac{1}{2}\right)(hh')'+XY\left(\log\frac{2}{\sqrt{X^2+Y^2}}+\frac{3}{2}+\frac{Y}{X}\tan^{-1}\frac{Y}{X}\right)\times$$

$$\times h''(s)+\frac{1}{2}XY\frac{d^3}{ds^3}\int_{-1}^{1}h(\sigma)\,\mathrm{sgn}\,(s-\sigma)\log|s-\sigma|\,d\sigma+$$

$$\left.+\frac{1}{4}Y\frac{d^3}{ds^3}\int_{-1}^{1}h^2(\sigma)\,\mathrm{sgn}\,(s-\sigma)\log|s-\sigma|\,d\sigma\right] \tag{1.21}$$

The first two terms agree with (1.16) as a check on matching.

1.5. Third inner approximation. We return once more to the inner expansion to find the first improvement on Prandtl's theory. The differential equation (1.5.1) would suggest that the next term in (1.13) is of order A^{-3}, but matching with (1.21) shows that a logarithmic term intervenes. Thus we continue the inner expansion as

$$\varphi \sim A^{-1}\Phi_1 + A^{-2}\Phi_2 + A^{-3}\log A\,\Phi_{32} + A^{-3}\Phi_{31} + \ldots \tag{1.22}$$

The notation is designed to indicate that the third approximation is properly regarded as consisting of the two terms in $A^{-3}\log A$ and A^{-3}, because for practical purposes the logarithm is of order unity.

As usual, the logarithmic term is much simpler to calculate than its algebraic companion. Substituting into (1.5) shows that Φ_{32} satisfies the two-dimensional Laplace equation and the conditions of zero velocity normal to the plate and finite velocity at the trailing edge. Matching with (1.21) gives the remaining boundary condition as

$$\Phi_{32} \sim \frac{1}{2}\alpha\, [Y\,(hh')' + XYh''] \tag{1.23}$$

The first term in the bracket implies, as before, a slight increase in the effective angle of attack. The second term represents curvature of the streamlines in the vicinity of the wing, induced by the trailing vortex system. This is nullified by adding the thin-airfoil solution for a parabolically cambered airfoil. Thus the coefficient of the logatithmic term is found to be

$$\begin{aligned}\Phi_{32} = \frac{1}{4}\alpha\ \mathrm{Im}\ [2\,(hh')'\,\{(Z^2 - h^2)^{1/2} - h\ \cosh^{-1}(Z/h)\} + \\ + h''\,\{Z\,(Z^2 - h^2)^{1/2} - h^2\ \cosh^{-1}(Z/h)\}]\end{aligned} \tag{1.24}$$

For the coefficient of the algebraic term, substituting (1.22) and (1.6) into (1.5.1) gives the differential equation

$$\begin{aligned}\Phi_{31_{XX}} + \Phi_{31_{YY}} = \alpha\ \mathrm{Im}\ [(hh')'\,(Z^2 - h^2)^{-1/2} - h^{-1}\,(hh')'\,Z\,(Z^2 - h^2)^{-1/2} + \\ + (hh')^2\,(Z^2 - h^2)^{-3/2} - hh'^2 Z\,(Z^2 - h^2)^{-3/2} + h''\ \cosh^{-1}(Z/h)]\end{aligned} \tag{1.25}$$

Introducing the conjugate complex variables $Z = X + iY$ and $\bar{Z} = X - iY$ facilitates finding the particular integral

$$\begin{aligned}\Phi_{31}^{(p)} = \frac{1}{4}\alpha\,\mathrm{Im}\,\bar{Z}\,[hh'^2\,(Z^2 - h^2)^{-1/2} - h'^2 Z\,(Z^2 - h^2)^{-1/2} - \\ - (2h'' + h'^2h^{-1})\,(Z^2 - h^2)^{1/2} + (hh')'\ \cosh^{-1}(Z/h) + h''Z\ \cosh^{-1}(Z/h)]\end{aligned} \tag{1.26.1}$$

A complementary solution of the homogeneous equation that preserves tangency, restores the Kutta-Zhukovskii condition, and provides matching with the last term in (1.21) is found by inspection to be

$$\begin{aligned}\Phi_{31}^{(c)} = \frac{\alpha}{4}\ \mathrm{Im}\Big[\Big\{\Big(\log\frac{4}{h} + \frac{3}{2}\Big)h'' + \frac{1}{2}\frac{d^3}{ds^3}\int_{-1}^{1} h\,(\sigma)\,\mathrm{sgn}\,(s - \sigma)\log|s - \sigma|\,d\sigma\Big\}\times \\ \times \{Z\,(Z^2 - h^2)^{1/2} - h^2\ \cosh^{-1}\frac{Z}{h}\} + \Big\{-2\frac{d}{ds}\int_{-1}^{1}\frac{h_2(\sigma)\,d\sigma}{s - \sigma} + \Big(2\log\frac{4}{h} + 1\Big)(hh')' - \\ - h'^2 + \frac{1}{2}\frac{d^3}{ds^3}\int_{-1}^{1} h^2(\sigma)\,\mathrm{sgn}\,(s - \sigma)\,\log|s - \sigma|\,d\sigma\Big\}\{(Z^2 - h^2)^{1/2} - \\ - h\ \cosh^{-1}\frac{Z}{h}\} - h''Z^2\cosh^{-1}\frac{Z}{h} - (hh')'Z\ \cosh^{-1}\frac{Z}{h} + h\Big(2hh'' + \frac{3}{2}h'^2\Big)\ \cosh^{-1}\frac{Z}{h}\Big]\end{aligned} \tag{1.26.2}$$

1.6. Third-order circulation and lift. The circulation Γ is $2\pi i$ times the coefficient of $\log\,(x^2 + y^2)^{1/2}$ or $\cosh^{-1}(Z/h)$ in the expansion for φ. Thus (1.22), (1.15), (1.24), (1.26.1) and (1.26.2) give

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{1}{2A}\int_{-1}^{1}\frac{h'(\sigma)\,d\sigma}{s-\sigma} + \frac{\log A}{4A^2}(2h'^2 + 3hh'') +$$
$$+\frac{1}{4A^2}\Big[\Big(2\log\frac{4}{h} - \frac{3}{2}\Big)h'^2 + \Big(3\log\frac{4}{h} + \frac{1}{2}\Big)hh'' - \tag{1.27}$$
$$-2\frac{d}{ds}\int_{-1}^{1}\frac{h_2(\sigma)\,d\sigma}{s-\sigma} + \frac{1}{2}h\frac{d^3}{ds^3}\int_{-1}^{1}h(\sigma)\,\mathrm{sgn}\,(s-\sigma)\log|s-\sigma|\,d\sigma +$$
$$+\frac{1}{2}\frac{d^3}{ds^3}\int_{-1}^{1}h^2(\sigma)\,\mathrm{sgn}\,(s-\sigma)\log|s-\sigma|\,d\sigma\Big] + \ldots$$

Here Γ_∞ is the two-dimensional value (1.9). A table for the first two integrals is given in [7]. The last two are conveniently calculated as the finite parts of divergent integrals

$$\frac{d}{ds}\int_{-1}^{1} g(\sigma)\,\mathrm{sgn}\,(s-\sigma)\log|s-\sigma|\,d\sigma = \tag{1.28}$$
$$= \mathrm{f.p.}\Big[\int_0^{1+s}\frac{g(s-\sigma)}{\sigma}d\sigma + \int_0^{1-s}\frac{g(s-\sigma)}{\sigma}d\sigma\Big]$$

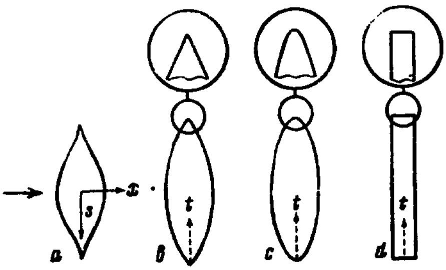


Fig. 2

The lift is related to the circulation by the Kutta-Zhukovskii law. Hence the lift-curve slope is related to (1.27) by

$$\frac{dC_L}{d\alpha} = 2\pi\int_0^1 h(s)\,\frac{\Gamma}{\Gamma_\infty}\,ds \tag{1.29}$$

2. Application to a family of planforms. We assess the utility of the foregoing formal analysis by applying it to the family of planforms shown in Fig. 2. These are described by

$$h(s) = k_n(1-s^2)^{1/2\,n} \qquad \Big(k_n = \Big[\int_0^1 (1-s^2)^{1/2\,n}\,ds\Big]^{-1}\Big) \tag{2.1.1}$$

The aspect ratio coefficient k_n for $n = 0,1,2,3$ is equal respectively

$$k_0 = 1, \qquad k_1 = \frac{4}{\pi}, \qquad k_2 = \frac{3}{2}, \qquad k_3 = \frac{16}{3\pi} \tag{2.12}$$

Because we anticipate complications at the blunter tips, we consider the smoothest shape first.

2.1. Cusped planform. The cusp-tipped wing of Fig.2a is described by

$$h(s) = \frac{16}{3\pi}(1-s^2)^{3/2} \tag{2.2}$$

Equation (1.27) gives the bound circulation as

$$\frac{\Gamma}{\Gamma_\infty} = 1 - 4A^{-1}(1-2s^2) - \frac{64}{\pi^2}A^{-2}\log A\,(1-s^2)(1-4s^2)\times \tag{2.3}$$
$$\times\frac{64}{3\pi^2}A^{-2}\Big[(1-s^2)\Big\{(1-2s^2)\log 2 - 2(1-5s^2)\log(1-s^2) -$$
$$-3(1-4s^2)\log\frac{3\pi}{4(1-s^2)^{3/2}} - \frac{1}{2}(1+7s^2) + 2(1-s^2)^{1/2}\times$$
$$\times(1-s\quad\sin^{-1}s)\Big\} + 4(1-16s^2-5s^4) + \frac{x^2}{8}(9-48s^2+40s^4)\Big]$$

This planform is so smooth that no complications arise from the tips, (2.3) being uniformly valid across the span.

2.2. Lenticular planform. For the lenticular wing of Fig. 2*b*, described by

$$h(s) = \tfrac{3}{2}(1-s^2) \tag{2.4}$$

the circulation is found to be

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{3}{2A}\left(2 - s\log\frac{1+s}{1-s}\right) - \frac{9}{8}\frac{\log A}{A^2}(3-7s^2) - \frac{9}{8A^2}\Big[\frac{1}{2}(1-3s^2)\times$$
$$\times\left(\log^2\frac{1+s}{1-s} - \pi^2\right) + (3-7s^2)\log\frac{8}{3\sqrt{1-s^2}} + 2s(3-s^2)\log\frac{1+s}{1-s} - \frac{35}{6} + \frac{31}{2}s^2\Big] \tag{2.5}$$

This result is not uniformly valid. It breaks down within an exponentially small distance of either tip, where the logarithm becomes so large that each term is of order unity. The difficulty arises from the fact that the flow at the tip does not become two-dimensional no matter how great A is. However, the singularity is so weak that (1.29) gives the correct lift-curve slope

$$\frac{dC_L}{d\alpha} = 2\pi\Big[1 - \frac{9}{4}A^{-1} - \frac{9}{5}A^{-2}\log A + \left(\frac{3}{20}x^2 - \frac{9}{5}\log\frac{8}{3} + \right.$$
$$\left. + \frac{27}{8}\log 2 - \frac{408}{25}\right)A^{-2} + \cdots\Big] \tag{2.6}$$

The non-uniformity could be corrected by constructing a third expansion, valid in the immediate vicinity of the tip, and matching it to the inner expansion. However, the result of this process can be deduced without carrying out the details. Because the region of non-uniformity is exponentially small, the tip must be magnified so much that to any order it resembles an infinite lifting triangle (Fig. 2*b*). There is then no characteristic length in the problem. The potential must therefore be a homogeneous function of the space coordinates; in spherical polar coordinates it will have the form

$$\varphi_{\text{tip}} \sim r^q f(\theta, \psi) \tag{2.7.1}$$

The exponent q will reduce to unity in the limit as $A \to \infty$, where the vertex angle of the triangle vanishes; for finite A it will have the form

$$q = 1 - \frac{c_1}{A} + \frac{c_2}{A^2} + \cdots \tag{2.7.2}$$

The circulation is proportional to one less than this power of the spanwise distance t from the tip

$$\Gamma_{\text{tip}} \propto t^{q-1} = s^{(q-1)\log t} \sim 1 - \left(\frac{c_1}{A} - \frac{c_2}{A^2}\right)\log t + \frac{1}{2}\left(\frac{c_1^2}{A^2} + \cdots\right)\log^2 t + \cdots \quad (2.8)$$

The last expansion, valid only far from the tip, can be used for matching with the inner solution. Introducing $t = 1 \pm s$ into (2.5) gives

$$\frac{\Gamma}{\Gamma_\infty} \sim 1 - \frac{3}{2A}\log t + \frac{9}{8}\frac{1}{A^2}\left[\log^2 t + 2(1 - \log 2)\log t\right] \quad (2.9)$$

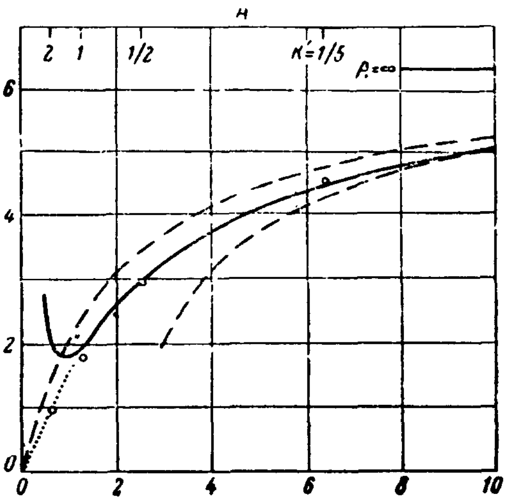


Fig. 3

and comparing with (2.8) shows that $c_1 / A = 3 / 2A$ (the ratio of root chord to span) and $c_2 = 9(1 - \log 2)/4$. Thus the circulation is rendered uniformly valid by extracting a factor

$$(1 - s^2)^{-c_1/A + c_2/A^2 + \cdots}$$

that accounts for all the logarithmically singular terms. This correction parallels that made for the sharp-edged two-dimensional airfoils and bodies of revolution in [8].

2.3. Elliptic planform. For the elliptic wing of Fig. 2c, described by

$$h(s) = \frac{4}{\pi}(1 - s^2)^{1/2} \quad (2.10)$$

the circulation is found to be

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{2}{A} - \frac{4}{\pi^2}\frac{\log A}{A^2}\frac{3 - 2s^2}{1 - s^2} + \frac{4}{\pi^2}\frac{1}{A^2}\left[\frac{5}{2} + \pi^2 - \log(1 - s^2) + \right.$$
$$\left. + \frac{\log 2}{1 - s^2} - \frac{3 - 2s^2}{1 - s^2}\log\frac{\pi}{\sqrt{1 - s^2}}\right] \quad (2.11)$$

This result evidently breaks down when the distance from the tip is of order A^{-2}, that is, of the order of the radius of curvature of the tip. However, the circulation is still integrable, and gives

$$\frac{dC_L}{d\alpha} = 2\pi\left[1 - \frac{2}{A} - \frac{16}{\pi^2}\frac{\log A}{A^2} + \frac{4}{\pi^2}\left(\frac{7}{2} + \pi^2 - 4\log\pi\right)\frac{1}{A^2}\right] \quad (2.12)$$

The first two terms are the expansion of Prandtl's famous result for the elliptic wing

$$\frac{dC_L}{d\alpha} = \frac{2\pi}{1 + 2/A} \quad (2.13)$$

This form has the advantage of vanishing at $A = 0$, though with twice the correct slope of slender-wing theory. Fig. 3 shows that our expanded form is the more accurate above $A = 5$.

We can recast our third approximation into a form analogous to Prandtl's. If we also replace the aspect ratio A by the ratio k' of minor to major axes according to $k' = 4/\pi A$, , Expression (2.12) assumes the simpler appearance

$$\frac{dC_L}{d\alpha} = 2\pi\left[1 + \frac{1}{2}\pi k' + \left(\log\frac{4}{k'} - \frac{7}{8}\right)k'^2\right]^{-1} \tag{2.14}$$

Krienes [9] has extracted an asymptotic result of this sort from his lifting-surface theory. However, he has $^{63}/_{128}$, in place of our $\frac{1}{2}$, and $\frac{7}{4}$ instead of $\frac{7}{8}$. His $^{63}/_{128}$ is inexplicable in view of Prandtl's result; the $\frac{7}{4}$ appears to be a slip, because our correction reduces the error, compared with Krienes' own lifting-surface result (Fig.3), from 13 to one percent at $k' = \frac{1}{2}$. Even for the circular wing (2.14) is only 12 percent high. The non-uniformity of the solution near the tip could again be corrected by constructing a supplementary expansion for that region. The tip problem would, in the first approximation, be that for a semi-infinite parabolic wing (Fig. 2c). This challenging potential problem has not yet been attempted.

2.4. Rectangular planform. The rectangular wing of Fig. 2d is best described by

$$h(s) = H(1 - s^2) \tag{2.15}$$

where H is the Heaviside step function, with the Dirac delta function as its derivative. Otherwise it is necessary to make the replacement

$$\int_{-1}^{1} \frac{h'(\sigma)\, d\sigma}{s - \sigma} \rightarrow \frac{d}{ds}\int_{-1}^{1} \frac{h(\sigma)\, d\sigma}{s - \sigma}$$

The first two terms of (1.27) give

$$\frac{\Gamma}{\Gamma_\infty} = 1 - \frac{1}{A}\frac{1}{1 - s^2} \tag{2.16}$$

but the subsequent terms contain singular functions and divergent integrals, and must be considered meaningless. Our solution now evidently breaks down at distances from the tip of the order of the chord length. The circulation itself behaves near the tip like

$$\frac{\Gamma}{\Gamma_\infty} \sim 1 - \frac{c}{4t} + \cdots \tag{2.17}$$

where c is the chord and t the distance from the tip. This cannot be integrated to find the lift.

These defects could be corrected by isolating the tip (Fig. 2d) and solving the problem of a lifting semi-infinite rectangular strip. Stewartson [10] has done this within the framework of Prandtl's lifting-line theory. Far from the tip his solution behaves like

$$\frac{\Gamma}{\Gamma_\infty} \sim 1 - \frac{c}{4t} + \left(\frac{c}{4t}\right)^2\left(\log\frac{4c}{t} + \gamma\right) + \cdots \qquad (\gamma = 0.5772\ldots) \tag{2.18}$$

in agreement with (2.17). Stewartson uses his result to calculate the lift of a finite rectangular wing. Probably the details would be altered by a lifting-surface treatment.

The foregoing analysis can be immediately extended to subsonic compressible flow using Göthert similarity rule, because of the linearizing assumptions. Calculation of nonlinear effects would require substantial additional analysis.

It would be of interest to extend the solution to the notorious problem of the swept wing. (A beginning was made in 1955 by Ciolkowski [11] at the suggestion of Friedrichs). Non-uniformity is to be anticipated at the root juncture, where the method of matched asymptotic expansions would require the lifting-surface solution for an infinite V-wing. Similar non-uniformities would appear at other planform discontinuities, deflected ailerons,etc.

Indeed, the three-dimensional lifting wing is a veritable treasure-trove of non-uniformities. One would treat more general airfoil sections by making the thin-airfoil approximation. This introduces further non-uniformities at leading and trailing edges, which can be corrected by still other local examinations [8]. Finally, the solution is not valid far downstream, where the vortex sheet rolls up [12].

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BIBLIOGRAPHY

1. Kaplun, S. and Lagerstrom, P.A., Asymptotic expansions of Navier-Stokes solutions for small Reynolds Number; Low Reynolds Number flow past a circular cylinder; Note on the preceding two papers. J. Math. Mech. Vol.5 and 6, 1957.

2. Lagerstrom, P.A. and Cole, J.D., Examples illustrating expansion procedures for the Navier-Stokes equations. J.Mech.Anal. 6 and 4, p.817-882, 1955.

3. Yakura, J.K., Theory of Entropy Layers and Nose Bluntness in Hypersonic Flow, in Hypersonic Flow Research. (F. Riddell, ed.), Academic, New York, pp.421-470, 1962.

4. Friedrichs, K.O., Special Topics in Fluid Dynamics, New York University, 1953.

5. Heaslet, M.A. and Lomax, H., The calculation of pressure on slender airplanes in subsonic and supersonic flow, U.S. N.A.C.A. Tech. Note 2900, 1953; Report 1185, 1954.

6. Prandtl, L., Tragflügeltheorie. 1. Mitteilung, in vier Abhandlungen zur Hydrodynamik und Aerodynamik, Keiser Wilhelm Inst. f. Strömungsforschung, Göttingen, 1927.

7. Van Dyke, M.D., Second-order subsonic airfoil-section theory and its practical application, U.S. N.A.C.A. Tech. Note 3390, 1955; Report 1274, 1956.

8. Van Dyke, M.D., Subsonic edges in thin-wing and slender-body theory, U.S. N.A.C.A. Tech. Note 3343, 1954.

9. Krienes, K., Die elliptische Tragfläche auf potential-theoretischer Grundlage, Z.a.M.M. 2, 20 (1940), 65-86; English Translation, N.A.C.A. Technical Memorandum 971.

10. Stewartson, K., A note on lifting line theory, Quartery Journal of Mechanics and Applied Mathematics 1, 13, 49-56, 1960.

11. Ciolkowski, S.I., The swept-back wing, Ph.D. Dissertation, New York University, 1955; University Microfilms № 24, 694, Ann.Arbor, Michigan.

12. Stern, M., The rolling up of a vortex sheet, Ph.D. Dissertation, New York University, 1954; University Microfilms № 22, 971.